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The Tracy–Widom Law for Some Sparse Random Matrices

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Abstract Consider the random matrix obtained from the adjacency matrix of a random d-regular graph by multiplying every entry by a random sign. The largest eigenvalue converges, after proper scaling, to the Tracy–Widom distribution.

Keywords Tracy–Widom distribution · Random graph · Edge of the spectrum · Airy point process

1 The Result

Let $G = (V, E)$ be an element of $\mathcal{G}(N, d)$, i.e. a random uniformly chosen d-regular graph with N vertices, and let $S : E \to \{-1, +1\}$ be independent random signs:

$$\mathbb{P}\big\{S(u, v) = 1\big\} = \mathbb{P}\big\{S(u, v) = -1\big\} = 1/2.$$

Consider the $N \times N$ (random) Hermitian matrix M,

$$M_{uv} = \begin{cases} S(u, v), & (u, v) \in E, \\ 0, & (u, v) \notin E \end{cases}$$

with eigenvalues $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_N$.

Theorem *Assume that M is as above, with $3 \le d \ll N^{2/3}$. The scaled largest eigenvalue*

$$2\left(\frac{d(d-1)}{(d-2)^2} N\right)^{2/3} \left\{\frac{\lambda_N}{2\sqrt{d-1}} - 1\right\} \tag{1}$$

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converges in distribution to the Tracy–Widom law TW_1. The same is true for the scaled smallest eigenvalue

$$-2\left(\frac{d(d-1)}{(d-2)^2}N\right)^{2/3}\left\{\frac{\lambda_1}{2\sqrt{d-1}}+1\right\}. \tag{2}$$

The proof is based on the combinatorial arguments from [2], which can be seen as a modification of Soshnikov's approach [9] to the asymptotic distribution of the extreme eigenvalues of random Hermitian matrices with independent entries. Another crucial ingredient is an estimate of McKay [4] on the number of subgraphs of a random graph.

2 Connection to Non-backtracking Walks

Consider the matrices

$$M^{(n)}=(d-1)^{n/2}\left\{U_n\left(\frac{M}{2\sqrt{d-1}}\right)-\frac{1}{d-1}U_{n-2}\left(\frac{M}{2\sqrt{d-1}}\right)\right\}, \tag{3}$$

where

$$U_n(\cos\theta)=\frac{\sin((n+1)\theta)}{\sin\theta}$$

are the Chebyshev polynomials of the second kind, and formally

$$U_{-2}\equiv U_{-1}\equiv 0.$$

It is not hard to see (cf. [8]) that, for any $u_0, u_n \in V$,

$$M^{(n)}_{u_0u_n}=\sum{}' M_{u_0u_1}M_{u_1u_2}\cdots M_{u_{n-1}u_n}, \tag{4}$$

where the sum is over all paths $p_n=u_0u_1u_2\cdots u_n$ such that

(a) $(u_j,u_{j+1})\in E$ for any $0\le j\le n-1$;
(b) $u_{j+2}\ne u_j$ for $0\le j\le n-2$ (p_n is *non-backtracking*).

Therefore

$$\operatorname{tr} M^{(n)}=\sum{}'' M_{u_0u_1}M_{u_1u_2}\cdots M_{u_{n-1}u_n},$$

where the sum is over paths p_n satisfying (a), (b), and

(c) $u_n=u_0$.

Taking the expectation (over S) and noting that $\mathbb{E}_S M^k_{uv}=((-1)^k+1)/2$, we deduce:

Lemma 1 *The expectation $\mathbb{E}_S \operatorname{tr} M^{(n)}$ is equal to the number $P_n(G)$ of paths p_n that satisfy* (a), (b), (c), *and*

(d) *every edge $(u,v)\in E$ appears an even number of times on p_n.*

In particular, $\mathbb{E}\operatorname{tr} M^{(2n+1)}=0$, we shall therefore study $\mathbb{E}\operatorname{tr} M^{(2n)}=\mathbb{E}_G P_{2n}(G)$, where $\mathbb{E}_G$ denotes expectation over the random choice of the graph G.

3 Diagrams

Following [2], we group the paths p_{2n} satisfying (a)–(d) into (topological) equivalence classes, which are in one-to-one correspondence with *diagrams*:

Definition 2 Let $\beta \in \{1, 2\}$.

- A *diagram* is an (undirected) multigraph $\bar{G} = (\bar{V}, \bar{E})$, together with a circuit $\bar{p} = \bar{u}_0 \bar{u}_1 \cdots \bar{u}_0$ on $\bar{G}$, such that
  - $\bar{p}$ is *non-backtracking* (meaning that no edge is followed by its reverse, unless the edge is a loop $\bar{u}\bar{u}$);
  - For every $(\bar{u}, \bar{v}) \in \bar{E}$,

$$\#\left\{j \mid \bar{u}_j = \bar{u},\ \bar{u}_{j+1} = \bar{v}\right\} + \#\left\{j \mid \bar{u}_j = \bar{v},\ \bar{u}_{j+1} = \bar{u}\right\} = 2;$$

  - the degree of $\bar{u}_0$ in $\bar{G}$ is 1; the degrees of all the other vertices are equal to 3.
- A *weighted diagram* is a diagram $\bar{G}$ together with a weight function $\bar{w} : \bar{E} \to \{-1, 0, 1, 2, \ldots\}$.

Informally, if $(u_1, u_2), (u_2, u_3), \ldots, (u_{k-1}, u_k)$ is a sequence of edges with

$$\deg u_1 = \deg u_k = 3, \qquad \deg u_2 = \cdots = \deg u_{k-1} = 2,$$

we replace $u_1 u_2 \cdots u_k$ with a single edge $\bar{e} = (\bar{u_1}, \bar{u_k})$, to which we assign the weight $\bar{w}(\bar{e}) = k - 2$. Thus, the path

1.2.3.4.5.6.7.4.6.5.7.4.3.2.1

corresponds to the diagram in Fig. 1 (left), whereas

1.2.3.4.5.6.7.8.9.7.8.9.6.5.10.11.12.13.14.15.16.12.13.14.15.16.12.11.10.5.4.3.2.1

corresponds to Fig. 1 (right).

More complicated paths (with vertices of degree greater than 3) also correspond to weighted diagrams; the formal correspondence is described in [2, II.1].

The following lemma summarises the results of [2, II.1–II.2] that we shall use:

Lemma 3

1. (Claim II.2.2) *All the diagrams can be classified according to a parameter* $s = 1, 2, \ldots$ (*see Fig.* 1). *A diagram with parameter* s *has* $\#\bar{V} = 2s$ *vertices and* $\#\bar{E} = 3s - 1$ *edges.*

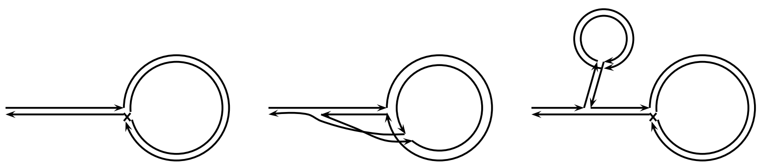

Fig. 1 Some diagrams: $s = 1$ (*left*), $s = 2$ (*center*, *right*)

2. (Proposition II.2.3) *The number* $D_1(s)$ *of diagrams corresponding to a given value* s *satisfies*

$$(s/C)^s \le D_1(s) \le C^{s-1} s^s,$$

where $C > 1$ *is a numerical constant.*

3. *For a diagram* δ *with parameter* s, *let* $W(\delta)$ *be the number of associated weighted diagrams, and* $\widetilde{W}(\delta)$*—the number of associated weighted diagrams with positive weights. Then*

$$\binom{n-3s}{3s-2} \le \widetilde{W}(\delta) \le W(\delta) \le \binom{n+3s-2}{3s-2}.$$

In particular, if $s = o(\sqrt{n})$,

$$\widetilde{W}(\delta),\, W(\delta) = (1+o(1))\frac{n^{3s-2}}{(3s-2)!}.$$

4. *Every path corresponding to* δ *has at most* $n-s+1$ *distinct vertices. If the associated weighted diagram has positive weights, the path has exactly* $n-s+1$ *distinct vertices; of these,* 1 *is of degree* 1, $n-3s+1$ *are of degree* 2, *and* $2s-1$ *are of degree* 3. *A general path is obtained from one as above by identifying several pairs of connected vertices, and erasing the connecting edges.*

5. (Claim II.1.4) *The number of different paths corresponding to a given weighted diagram is at most* N^{n-s+1}. *If the weights are positive, the number is exactly*

$$N(N-1)\cdots(N-n+s).$$

4 Counting the Paths

Fix a diagram δ with parameter s. We shall now estimate the expected number of paths p_{2n} associated to δ in a random graph G. We shall use the following special case of [4, Theorem 2.10].

Let L be a subgraph of the complete (labelled) graph K_N; let ℓ_u be the degree of u in L, $E_G = \#E(G) = dN/2$, and $E_L = \#E(L) = \sum \ell_u/2$. Finally, denote $a^{[b]} = a!/(a-b)!$

Proposition 4 (McKay) *If* $E_G - E_L \ge 3d(d+1)$, *then the probability* F_L *that a random* d*-regular graph* G *contains* L *satisfies*

$$\xi(L)\frac{\prod_u d^{[\ell_u]}}{2^{E_L} E_G^{[E_L]}} \le F_L \le \Xi(L)\frac{\prod_u d^{[\ell_u]}}{2^{E_L} E_G^{[E_L]}},$$

where

$$\xi(L) = \left\{ \frac{1 - \frac{d(d+1)}{2(E_G - E_L - 2d(d+1))}}{1 + \frac{d^2}{2(E_G - 2d^2 - \frac{e-1}{e} E_L)}} \right\}^{E_L} \frac{E_G^{[E_L]}}{(E_G-1)^{[E_L]}},$$

$$\Xi(L) = \frac{E_G^{[E_L]}}{(E_G - 2d^2)^{[E_L]}}.$$

We shall deduce the following estimates:

Lemma 5

1. *For* $n \le N$,

$$\mathbb{E} P_{2n}(G) \le n(d-2)(d-1)^{n-1} \exp\left\{ \frac{C n^{3/2}}{N^{1/2}} (1 + d/\sqrt{nN}) \right\}, \tag{5}$$

where $C > 0$ *is a numerical constant.*

2. *For* $d^2/N \ll n \ll \min(\sqrt{n}, N/d)$,

$$\mathbb{E} P_{2n}(G) = (1 + o(1))\,(d-1)^n \sum_{s \ge 1} \frac{(d-2)^{2s-1}}{d^{s-1}(d-1)^s N^{s-1}} \frac{n^{3s-2}}{(3s-2)!} D_1(s). \tag{6}$$

Proof Let p_{2n} be a path associated to a diagram δ with parameter s, and let L be the induced (undirected) graph:

$$E(L) = \{(u_j, u_{j+1})\}.$$

Then, by item 4 of Lemma 3,

$$\begin{aligned} \frac{\prod_u d^{[\ell_u]}}{2^{E_L} E_G^{[E_L]}} &\le \frac{d^{n-s+1}(d-1)^{n-s}(d-2)^{2s-1}}{2^n (dN/2)^{[n]}} \\ &= \frac{(d-2)^{2s-1}}{d^{s-1}(d-1)^s} \left(\frac{d-1}{N}\right)^n \frac{(dN/2)^n}{(dN/2)^{[n]}} \\ &\le \exp\left\{\frac{C_1 n^2}{dN}\right\} \left(\frac{d-1}{N}\right)^n \frac{(d-2)^{2s-1}}{d^{s-1}(d-1)^s}, \end{aligned} \tag{7}$$

where $C_1 > 0$ is a numerical constant. Also,

$$\Xi(L) = \frac{(dN/2)^{[n]}}{(dN/2 - 2d^2)^{[n]}} \le \exp\{C_2 nd/N\};$$

therefore

$$F_L \le \exp\left\{\frac{C_1 n^2}{dN} + \frac{C_2 nd}{N}\right\} \left(\frac{d-1}{N}\right)^n \frac{(d-2)^{2s-1}}{d^{s-1}(d-1)^s}.$$

According to item 3, the number of ways to choose the weights on δ is at most

$$\binom{n+3s-2}{3s-2} \le \frac{n^{3s-2}}{(3s-2)!},$$

and, according to item 4, the number of ways to choose the vertices is at most N^{n-s+1}. Finally, the number of diagrams with parameter s is at most $C^{s-1}s^s$. Therefore

$$\begin{aligned}
\mathbb{E}P_{2n}(G) &\le \sum_{s\ge 1} C^{s-1} s^s N^{n-s+1} \frac{n^{3s-2}}{(3s-2)!} \\
&\quad \times \exp\left\{\frac{C_1 n^2}{dN} + \frac{C_2 nd}{N}\right\} \left(\frac{d-1}{N}\right)^n \frac{(d-2)^{2s-1}}{d^{s-1}(d-1)^s} \\
&\le n(d-2)(d-1)^{n-1} \exp\left\{\frac{C_1 n^2}{dN} + \frac{C_2 nd}{N}\right\} \\
&\quad \times \sum_{s\ge 1} \frac{(C_3 n^3/N)^{s-1}}{(2(s-1))!} \\
&\le n(d-2)(d-1)^{n-1} \exp\left\{\frac{C_4 n^{3/2}}{N^{1/2}}(1 + d/\sqrt{nN})\right\}. \qquad (8)
\end{aligned}$$

This proves the first statement.

Suppose $d^2/N \ll n \ll \min(\sqrt{N}, N/d)$. Choose s_0 such that

$$\begin{cases} s_0 = 1, & n \le N^{1/4}, \\ n^{3/2}/N^{1/2} \ll s_0 \ll \sqrt{n}, & n > N^{1/4}. \end{cases}$$

For $s \le s_0$ and positive weights, the inequalities above are asymptotic equalities. Namely,

$$\begin{aligned}
F_L &= (1+o(1)) \left(\frac{d-1}{N}\right)^n \frac{(d-2)^{2s-1}}{d^{s-1}(d-1)^s}, \\
\widetilde{W}(\delta) &= (1+o(1)) \frac{n^{3s-2}}{(3s-2)!}.
\end{aligned}$$

Hence the contribution of paths with positive weights and $s \le s_0$ is

$$(1+o(1)) \sum_{1\le s\le s_0} D_1(s) N^{n-s+1} \frac{n^{3s-2}}{(3s-2)!} \left(\frac{d-1}{N}\right)^n \frac{(d-2)^{2s-1}}{d^{s-1}(d-1)^s}.$$

The contribution of paths with $s \le s_0$ and not all weights positive is negligible, since

$$W(\delta) - \widetilde{W}(\delta) = o(1) \frac{n^{3s-2}}{(3s-2)!}.$$

The contribution of paths with $s > s_0$ is also negligible, by the estimates in the proof of item 1 of this lemma (see (8) above). Thus

$$\begin{aligned}
\mathbb{E}P_{2n}(G) &= (1+o(1)) \sum_{1\le s\le s_0} D_1(s) N^{n-s+1} \frac{n^{3s-2}}{(3s-2)!} \left(\frac{d-1}{N}\right)^n \frac{(d-2)^{2s-1}}{d^{s-1}(d-1)^s} \\
&= (1+o(1)) \sum_{1\le s<\infty} D_1(s) N^{n-s+1} \frac{n^{3s-2}}{(3s-2)!} \left(\frac{d-1}{N}\right)^n \frac{(d-2)^{2s-1}}{d^{s-1}(d-1)^s}. \qquad \square
\end{aligned}$$

5 Conclusion of the Proof

Applying Lemma 1 and using the definition of $M^{(2n)}$, we immediately deduce:

Corollary 6

1. *For* $n \leq N$,

$$\mathbb{E}\,\mathrm{tr}\,U_{2n}\left\{\frac{M}{2\sqrt{d-1}}\right\} \leq n \exp\left\{\frac{C_4 n^{3/2}}{N^{1/2}}(1+d/\sqrt{nN})\right\},$$

where $C > 0$ *is a numerical constant.*

2. *For* $d^2/N \ll n \ll \min(\sqrt{N}, N/d)$,

$$\mathbb{E}\,\mathrm{tr}\,U_{2n}\left\{\frac{M}{2\sqrt{d-1}}\right\} = (1+o(1))\,n\sum_{s\geq 1}\left(\frac{(d-2)^2 n^3}{d(d-1)N}\right)^{s-1}\frac{D_1(s)}{(3s-2)!}.$$

Now, by [2, Theorem I.5.3, Proposition II.2.5], an $N' \times N'$ random matrix A_{inv} from the Gaussian Orthogonal Ensemble GOE(N') satisfies, for $n = o(\sqrt{N'})$,

$$\mathbb{E}U_{2n}\left(A_{\mathrm{inv}}/(2\sqrt{N'-2})\right) = (1+o(1))\,n\sum_{s\geq 1}(n^3/N')^{s-1}\frac{D_1(s)}{(3s-2)!}.$$

Hence, for $N' = \lfloor\frac{d(d-1)}{(d-2)^2}N\rfloor$, the second item of Corollary 6 implies:

$$\mathbb{E}\,\mathrm{tr}\,U_{2n}\left\{\frac{M}{2\sqrt{d-1}}\right\} = (1+o(1))\mathbb{E}\,\mathrm{tr}\,U_{2n}\left\{\frac{A_{\mathrm{inv}}}{2\sqrt{N-2}}\right\}.$$

Applying the arguments of [2, II.3], this is extended to expectations of products of several such traces.

As in [2, I.5] this implies that the random point process

$$\left\{2\left(\frac{d(d-1)}{(d-2)^2}N\right)^{2/3}\left[\frac{\lambda_j}{2\sqrt{d-1}}-1\right]\right\}_{j=1}^{N}$$

converges (in the appropriate topology, [2, I.2]) to the Airy point process $\mathfrak{Ai}_1$, and hence the distribution of (1) converges to TW_1.

6 Some Remarks

1. This note continues the study of the distribution of the scaled largest eigenvalues of random matrices with independent entries. For the Gaussian Orthogonal Ensemble, the limiting distribution was identified by Tracy and Widom, building on several earlier works; see [10] and references therein. This result was extended to a rather general class of matrices with independent entries by Soshnikov [9]. Important further results have been obtained by Ruzmaikina [7], and Khorunzhiy and Vengerovsky [3]. A different proof of Soshnikov's theorem was given in [2]; it is the basis of the present argument.
2. In the physical literature, the spectrum of random matrices similar to M has been recently studied by Oren, Godel, and Smilansky [6], who have focused on the bulk of the spectrum (rather than the extreme eigenvalues).

3. The conclusion of the theorem remains valid for more general weights $S(u, v)$, such that
   (A1) The distribution of every $S(u, v)$ is symmetric;
   (A2) $\mathbb{E}S(u, v)^{2k} \le (C_0 k)^k$;
   (A3$_1$) $\mathbb{E}S(u, v)^2 = 1$.
   The extension of the proof follows the lines of [2, Part III].
4. One may also consider complex weights $S(u, v)$ that satisfy (A1), (A2), and
   (A3$_2$) $\mathbb{E}(\Re S(u, v))^2 = \mathbb{E}(\Im S(u, v))^2 = 1/2$, $\mathbb{E}(\Re S(u, v) \Im S(u, v)) = 0$
   (for example, $S(u, v)$ may be uniformly distributed on the unit circle.) Then the scaled extreme eigenvalues (1), (2) converge in distribution to the Tracy–Widom law TW_2.
5. If all the weights $S(u, v)$ are equal to 1, the matrix M is the adjacency matrix A_G of G. Obviously, the largest eigenvalue of A_G is equal to d. Numerical evidence due to Miller, Novikoff, and Sabelli [5] indicates that the second largest eigenvalue λ_2 of A_G converges to TW_1 after proper scaling. If true, this would have important consequences, see [5] and references therein.
6. Our result can be reformulated as a bound for the "new" eigenvalues of a random 2-lift of a random d-regular graph. We refer the reader to the work of Bilu and Linial [1] for the definitions.

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